Pisier's Self-Dual Hilbert Operator Space

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Abstract

Here we present Gilles Pisier's remarkable result that given any Hilbert space $\mathscr{H}$ there exists a unique operator space structure on $\mathscr{H}$ such that the conjugate operator Hilbert space $\overline{\mathscr{H}}$ and the dual operator Hilbert space $\mathscr{H}^*$ are completely isometric. We give the results as presented by Effros and Ruan [ER00], and the interested reader may reference Pisier's original work [Pis96] for more details and properties of OH.

1 Pisier's Self-Dual Operator Hilbert Space

As is well known, given any Hilbert space $\mathscr{H}$ we may consider both the conjugate operator Hilbert space $\overline{\mathscr{H}}$, and its dual operator Hilbert space $\mathscr{H}^*$. The map $\theta : \overline{H} \longrightarrow \mathscr{H}^*, \overline{\xi} \mapsto f_\xi, f_\xi(\zeta) := (\zeta | \, \xi)$, is an isometry by the Riesz representation theorem. It was shown by Pisier that there is a unique operator space structure on $\mathscr{H}$ so that θ becomes a complete isometry. Given two Hilbert spaces $\mathscr{H}, \mathscr{K}$, define the operator

$$V : \mathscr{H} \otimes_2 \overline{\mathscr{K}} \longrightarrow \mathcal{HS}(\mathscr{K}, \mathscr{H}), \eta \otimes \overline{\xi} \mapsto x_{\eta\otimes\overline{\xi}}, x_{\eta\otimes\overline{\xi}}(\zeta) = (\zeta | \, \xi)\, \eta.$$

Letting $(e_s)_{s\in\mathfrak{S}}$ be an orthonormal basis for $\mathscr{K}$.

$$\left\| x_{\eta\otimes\overline{\xi}} \right\|_2 = \left(\text{trace}\, x^*_{\eta\otimes\overline{\xi}} x_{\eta\otimes\overline{\xi}}\right)^{\frac{1}{2}} = \left(\sum_{s\in\mathfrak{S}} \left(x_{\eta\otimes\overline{\xi}} e_s \middle| \, x_{\eta\otimes\overline{\xi}} e_s \right) \right)^{\frac{1}{2}} = \left(\sum_{s\in\mathfrak{S}} \left| \left(e_s | \, \xi \right) \right|^2 \|\eta\|^2 \right)^{\frac{1}{2}} = \|\xi\| \, \|\eta\| \,,$$

where the last equality is by Parseval's equality. Thus, given any $\eta \in \mathscr{H}, \overline{\xi} \in \overline{\mathscr{K}}$, we have $x_{\eta\otimes\overline{\xi}}$ is a Hilbert-Schmidt operator from $\mathscr{K}$ to $\mathscr{H}$, and this has shown that the Hilbertian tensor product of the two Hilbert spaces is isometric to the Hilbert-Schmidt operators between the respective spaces. By restricting the codomain of V to $V(\mathscr{H} \otimes_2 \overline{\mathscr{K}})$ we have that V is a unitary, and thus have the unitary equivalence

$$\sigma : \mathscr{B}(\mathscr{H} \otimes_2 \overline{\mathscr{K}}) \longrightarrow \mathscr{B}(\mathcal{HS}(\mathscr{K}, \mathscr{H})), u \mapsto VuV^{-1}.$$

Given $b \in \mathscr{B}(\mathscr{H})$, and $a \in \mathscr{B}(\mathscr{K})$, we have $\sigma(b \otimes \overline{a})x = bxa^*$. To see this suppose that $x = x_{\eta\otimes\overline{\xi}}$. It then follows

$$\sigma(b \otimes \overline{a})x_{\eta\otimes\overline{\xi}} = V(b \otimes \overline{a})V^{-1}x_{\eta\otimes\overline{\xi}} = V(b \otimes \overline{a})(\eta \otimes \overline{\xi}) = V(b\eta \otimes \overline{a\xi}) = x_{b\eta\otimes\overline{a\xi}} = bx_{\eta\otimes\overline{\xi}}a^*.$$

To see this note that for any $\zeta \in \mathscr{K}$ we have

$$x_{b\eta\otimes\overline{a\xi}}(\zeta) = (\zeta | \, a\xi)\, b\eta = b(\left(a^*\zeta \middle| \, \xi\right) \eta) = bx_{\eta\otimes\overline{\xi}}a^*\zeta.$$

This then shows that for any $u = \sum b_i \otimes \overline{a_i} \in \mathscr{B}(\mathscr{H}) \otimes \mathscr{B}(\overline{\mathscr{K}})$,

$$\|u\| = \sup\left\{ \left\| \sum b_i x a_i^* \right\|_2 : \|x\|_2 \leq 1 \right\}.$$

Now, for Hilbert space $\mathscr{H}$ with sesquilinear form $(\cdot|\cdot)$, we have an induced *matrix sesquilinear form* given by

$$M_m(\mathscr{H}) \times M_n(\mathscr{H}) \longrightarrow M_m \otimes M_n, (\eta, \xi) \mapsto ((\eta|\xi)) = \left[\left(\eta_{kl} \middle| \xi_{ij}\right) \right]_{i,j,k,l}.$$

Given $\xi, \eta \in M_n(\mathscr{H})$, let

$$\begin{aligned} \eta &= \sum \beta^{(h)} \otimes e_h \\ \xi &= \sum \alpha^{(h)} \otimes e_h, \end{aligned}$$

where $(e_h) \subset \mathscr{H}$ is an orthonormal basis. Then we have

$$\begin{aligned} ((\eta|\xi)) &= \left[\left(\eta_{kl} \middle| \xi_{ij}\right) \right] \\ &= \left[\left(\sum \beta_{kl}^{(h)} \otimes e_h \middle| \sum \alpha_{ij}^{(h)} \otimes e_h \right) \right] \\ &= \left[\sum \left(\overline{\alpha_{ij}^{(h)}} \beta_{kl}^{(h)} e_h \middle| e_h \right) \right] \\ &= \left[\sum \beta^{(h)} \otimes \overline{\alpha^{(h)}} \right] \in M_n \otimes \overline{M_n}. \end{aligned}$$

At this point we are then able to prove an analogue to the Schwarz inequality.

Theorem 1.1 (Haagerup [Haa85])**.** *Given a Hilbert space $\mathscr{H}$ and $n \in \mathbb{N}$, then for any $\xi, \eta \in M_n(\mathscr{H})$, we have*

$$\|((\eta|\xi))\| \le \|((\eta|\eta))\|^{\frac{1}{2}} \|((\xi|\xi))\|^{\frac{1}{2}}.$$

Proof. Given $\xi, \eta \in M_n(\mathscr{H})$,

$$\begin{aligned} \eta &= \sum \beta^{(h)} \otimes e_h \\ \xi &= \sum \alpha^{(h)} \otimes e_h \end{aligned}$$

then we may assume that $\mathscr{H} = \mathscr{K} = \mathbb{C}^n$, since

$$((\eta|\xi)) = \sum \beta^{(h)} \otimes \overline{\alpha^{(h)}} \in M_n \otimes \overline{M_n}.$$

Furthermore recall that if $\alpha, \beta \in M_n$ then

$$\operatorname{trace}(\beta\alpha^*) = \operatorname{trace}\left(\left[\sum_k \beta_{ik} \overline{\alpha_{jk}} \right] \right) = \sum_{i,j} \beta_{ij} \overline{\alpha_{ij}} = \sum_{i,j} \left(\beta_{ij} \middle| \alpha_{ij} \right).$$

We then have by our above calculations that

$$\begin{aligned} \|((\eta|\xi))\| &= \left\| \sum \beta^{(h)} \otimes \overline{\alpha^{(h)}} \right\| \\ &= \sup \left\{ \left\| \sum \beta^{(h)} x \alpha^{(h)*} \right\|_2 : \|x\|_2 \le 1 \right\} \\ &= \sup \left\{ \left| \operatorname{trace}\left(\sum \beta^{(h)} x \alpha^{(h)*} y^* \right) \right| : \|x\|_2, \|y\|_2 \le 1 \right\}. \end{aligned}$$

Thus, fix an x and y as above and let $x = v|x|, y = w|y|$ be the corresponding polar decompositions. Write $x = x_1 x_2, x_1 = v|x|^{\frac{1}{2}}, x_2 = |x|^{\frac{1}{2}}$, and take an analogous decomposition for y. It then follows that $x_1 x_1^* = |x^*|, x_2^* x_2 = |x|$, where the first equality follows since

$$\left|x^*\right|^2 = xx^* = v|x||x|v^* = v|x|v^*v|x|v^* = (v|x|v^*)^2.$$

The analogous equalities also hold for y. We then have the following string of inequalities

$$\begin{aligned}\left|\operatorname{trace}\left(\sum \beta^{(h)} x \alpha^{(h)*} y^*\right)\right| &\leq \sum \left|\operatorname{trace}\left(\beta^{(h)} x_1 x_2 \alpha^{(h)*} y_2^* y_1^*\right)\right| \\ &= \sum \left|\operatorname{trace}\left((y_1^* \beta^{(h)} x_1)(y_2 \alpha^{(h)} x_2^*)^*\right)\right| \\ &\leq \sum \left(\operatorname{trace}\left(\left(y_1^* \beta^{(h)} x_1\right)\left(y_1^* \beta^{(h)} x_1\right)^*\right)\right)^{\frac{1}{2}} \left(\operatorname{trace}\left(\left(y_2 \alpha^{(h)} x_2^*\right)\left(y_2 \alpha^{(h)} x_2^*\right)^*\right)\right)^{\frac{1}{2}}\end{aligned}$$

The equality holds by decomposition of x and y into $x_1 x_2$ and $y_1 y_2$, respectively, and then using the tracial property, and the last inequality follows by applying the classical Cauchy-Schwarz inequality. Continuing we get

$$\begin{aligned}&\leq \left(\sum \operatorname{trace}\left(\left(y_1^* \beta^{(h)} x_1\right)\left(y_1^* \beta^{(h)} x_1\right)^*\right)\right)^{\frac{1}{2}} \left(\sum \operatorname{trace}\left(\left(y_2 \alpha^{(h)} x_2^*\right)\left(y_2 \alpha^{(h)} x_2^*\right)^*\right)\right)^{\frac{1}{2}} \\ &= \left\|\sum y_1^* \beta^{(h)} x_1\right\|_2 \left\|\sum y_2 \alpha^{(h)} x_2^*\right\|_2 \\ &= \left\|\left|\sum y_1^* \beta^{(h)} x_1\right|\right\|_2 \left\|\left|\sum y_2 \alpha^{(h)} x_2^*\right|\right\|_2 \\ &= \left|\sum \operatorname{trace}\left(y_1^* \beta^{(h)} x_1 x_1^* \beta^{(h)*} y_1\right)\right|^{\frac{1}{2}} \left|\sum \operatorname{trace}\left(y_2 \alpha^{(h)} x_2^* x_2 \alpha^{(h)*} y_2^*\right)\right|^{\frac{1}{2}} \\ &= \left|\sum \operatorname{trace}\left(\beta^{(h)} \left|x^*\right| \beta^{(h)*} \left|y^*\right|\right)\right|^{\frac{1}{2}} \left|\sum \operatorname{trace}\left(\alpha^{(h)} |x| \, \alpha^{(h)*} |y|\right)\right|^{\frac{1}{2}} \\ &= \left|\left(\sigma\left(\sum \beta^{(h)} \otimes \overline{\beta^{(h)}}\right) \left|x^*\right| \middle| \left|y^*\right|\right)\right|^{\frac{1}{2}} \left|\left(\sigma\left(\sum \alpha^{(h)} \otimes \overline{\alpha^{(h)}}\right) |x| \middle| |y|\right)\right|^{\frac{1}{2}} \\ &\leq \left\|\sum \beta^{(h)} \otimes \overline{\beta^{(h)}}\right\|^{\frac{1}{2}} \left\|\sum \alpha^{(h)} \otimes \overline{\alpha^{(h)}}\right\|^{\frac{1}{2}}.\end{aligned}$$

□

Given a Hilbert space $\mathscr{H}$ and $n \in \mathbb{N}$, we define the *OH matrix norm* $\|\cdot\|_{o,n}$ by

$$\|\xi\|_{o,n} := \|((\xi|\,\xi))\|^{\frac{1}{2}}, \xi \in M_n(\mathscr{H}).$$

Thus, we have that if $(e_h)_h \subset \mathscr{H}$ is an orthornormal basis and $\xi = \sum \alpha^{(h)} \otimes e_h$, that

$$\|\xi\|_{o,n} = \left(\left\|\sum \alpha^{(h)} \otimes \overline{\alpha^{(h)}}\right\|\right)^{\frac{1}{2}} = \left(\sup\left\{\left|\operatorname{trace}\left(\sum \alpha^{(h)} x \alpha^{(h)*} y^*\right)\right| : x, y \in B_{HS(\mathscr{H},\mathscr{H})}\right\}\right)^{\frac{1}{2}}.$$

We will now prove that $\|\cdot\|_o$ satisfies Ruan's axioms and furthermore that it induces the unique operator space structure on the Hilbert space such that its conjugate operator space and dual operator space are completely isometric.

Theorem 1.2 (Pisier)**.** *Given a Hilbert space $\mathscr{H}$, then the OH matrix norm on $\mathscr{H}$ satisfies Ruan's axioms, and we let the corresponding operator space $\left(\mathscr{H}, \left\{\|\cdot\|_{o,n}\right\}_{n\in\mathbb{N}}\right)$ be denoted by $\mathscr{H}_o$. Letting $\overline{\mathscr{H}}$ and $\mathscr{H}^*$ have the induced conjugate and dual operator space structures, respectively, then $\|\cdot\|_o$ is the unique operator space matrix norm for which the corresponding mapping*

$$\psi : \overline{\mathscr{H}} \longrightarrow \mathscr{H}^*$$

is completely isometric.

Proof. We begin by proving that $\|\cdot\|_o$ satisfies Ruan's axioms. Let $(e_h)_h \subset \mathscr{H}$ be an orthonormal basis for $\mathscr{H}$ and let $\xi \in M_n(\mathscr{H}), \eta \in M_m(\mathscr{H})$. Then we have that for

$$\begin{aligned}\xi &= \sum \alpha^{(h)} \otimes e_h \\ \eta &= \sum \beta^{(h)} \otimes e_h,\end{aligned}$$

that

$$
\begin{aligned}
((\eta \oplus \xi | \, \eta \oplus \xi)) &= \sum_h (\beta^{(h)} \oplus \alpha^{(h)}) \otimes (\overline{\beta^{(h)}} \oplus \overline{\alpha^{(h)}}) \\
&= \sum_h (\beta^{(h)} \otimes \overline{\beta^{(h)}}) \oplus (\beta^{(h)} \otimes \overline{\alpha^{(h)}}) \oplus (\alpha^{(h)} \otimes \overline{\beta^{(h)}}) \oplus (\alpha^{(h)} \otimes \overline{\alpha^{(h)}}) \\
&= ((\eta | \, \eta)) \oplus ((\eta | \, \xi)) \oplus ((\xi | \, \eta)) \oplus ((\xi | \, \xi))
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\|\eta \oplus \xi\|_o^2 &= \|((\eta \oplus \xi | \, \eta \oplus \xi))\| \\
&= \|((\eta | \, \eta)) \oplus ((\eta | \, \xi)) \oplus ((\xi | \, \eta)) \oplus ((\xi | \, \xi))\| \\
&\leq \max \left\{ \|\eta\|_o^2 , \|\xi\|_o^2 , \|\eta\|_o , \|\xi\|_o \right\} \\
&= \max \|\eta\|_o^2 , \|\xi\|_o^2 .
\end{aligned}
$$

We therefore have that $\|\eta \oplus \xi\|_o \leq \max \left\{ \|\eta\|_o , \|\xi\|_o \right\}$.

Let $\xi = \sum_h \xi^{(h)} \otimes e_h \in M_p(\mathscr{H})$, $\alpha \in M_{n,p}$ and $\beta \in M_{p,n}$. We first show that

$$
((\alpha\xi\beta | \, \alpha\xi\beta)) = (\alpha \otimes \overline{\alpha}) \, ((\xi | \, \xi)) \, (\beta \otimes \overline{\beta}).
$$

First note that

$$
\alpha\xi\beta = \left[\sum_{k,l} \alpha_{ik} \xi_{kl}^{(h)} \beta_{kj} \right]_{i,j} \in M_n(\mathscr{H}),
$$

and thus we will we have

$$
\begin{aligned}
((\alpha\xi\beta | \, \alpha\xi\beta)) &= \sum_h \alpha\xi^{(h)}\beta \otimes \overline{\alpha\xi^{(h)}\beta} \\
&= \left[\sum_h \left(\alpha\xi^{(h)}\beta \right)_{kl} \left(\overline{\alpha\xi^{(h)}\beta} \right)_{ij} \right]_{i,j,k,l} \\
&= \left[\sum_h \left(\sum_{a,b} \alpha_{ka} \xi_{ab}^{(h)} \beta_{bl} \right) \left(\sum_{c,d} \overline{\alpha_{ic}} \overline{\xi_{cd}^{(h)}} \overline{\beta_{dj}} \right) \right]_{i,j,k,l} \\
&= \left[\sum_{a,b,c,d,h} \alpha_{ka} \xi_{ab}^{(h)} \beta_{bl} \overline{\alpha_{ic}} \overline{\xi_{cd}^{(h)}} \overline{\beta_{dj}} \right]_{i,j,k,l} \\
&= \left[\sum_{a,b,c,d,h} \alpha_{ka} \overline{\alpha_{ic}} \xi_{ab}^{(h)} \overline{\xi_{cd}^{(h)}} \beta_{bl} \overline{\beta_{dj}} \right]_{i,j,k,l} \\
&= \left[\sum_{a,b,c,d} \alpha_{ka} \overline{\alpha_{ic}} \left(\sum_h \xi_{ab}^{(h)} \overline{\xi_{cd}^{(h)}} \right) \beta_{bl} \overline{\beta_{dj}} \right]_{i,j,k,l} \\
&= (\alpha \otimes \overline{\alpha}) \, ((\xi | \, \xi)) \, (\beta \otimes \overline{\beta}).
\end{aligned}
$$

Since we have that

$$
\begin{aligned}
\left\| \alpha \otimes \overline{\alpha} \right\| &= \|\alpha\| \left\| \overline{\alpha} \right\| = \|\alpha\|^2 , \\
\left\| \beta \otimes \overline{\beta} \right\| &= \|\beta\| \left\| \overline{\beta} \right\| = \|\beta\|^2 ,
\end{aligned}
$$

we then have the inequality

$$
\|\alpha\xi\beta\|_o^2 = \|((\alpha\xi\beta | \, \alpha\xi\beta))\| \leq \|\alpha\|^2 \, \|\xi\|_o^2 \, \|\beta\|^2 ,
$$

which gives us Ruan's second axiom. Therefore, $\|\cdot\|_o$ is indeed an operator space matrix norm.

We now prove that $\psi : \overline{\mathscr{H}_o} \longrightarrow \mathscr{H}_o^*$ is a complete isometry. Given $n \in \mathbb{N}$, and $\bar{\xi} \in M_n(\overline{\mathscr{H}_o})$, we have

$$\psi^{(n)}(\bar{\xi}) \in M_n(\mathscr{H}_o^*) \cong \mathscr{CB}(\mathscr{H}_o, M_n),$$

and thus we set $\varphi = \psi^{(n)}(\bar{\xi})$. Therefore for $m \in \mathbb{M}$, and $\eta \in M_m(\mathscr{H}_o)$ we see

$$\varphi^{(m)}(\eta) = \left[\varphi(\eta_{kl})\right]_{k,l} = \left[\psi^{(n)}(\bar{\xi})(\eta_{kl})\right]_{k,l} = \left[\psi(\overline{\xi_{ij}})(\eta_{kl})\right]_{i,j,k,l} = \left[\left(\eta_{kl} \middle| \, \xi_{ij}\right)\right]_{i,j,k,l} = ((\eta | \, \xi)) \, .$$

It then follows by applying our analogue of Cauchy-Schwarz that

$$\left\|\varphi^{(m)}(\eta)\right\| \leq \|\eta\|_o \, \|\xi\|_o \, .$$

Thus, we have

$$\left\|\psi^{(n)}(\bar{\xi})\right\|_{cb} \leq \|\xi\|_o \, .$$

Conversely, letting $\eta = \frac{\xi}{\|\xi\|_o}$, implies

$$\left\|\psi^{(n)}(\bar{\xi})(\eta)\right\|_{cb} = \|((\xi | \, \xi))\| \, \frac{1}{\|\xi\|_o} = \|\xi\|_o \, ,$$

implying that ψ is a complete isometry.

Suppose now that $\|\cdot\|'$ is another operator space matrix norm such that ψ is a complete isometry. Then by our above calculations we have

$$\|\xi\|_o = \sup\left\{\|((\eta | \, \xi))\| \, : \, \|\eta\|' \leq 1\right\} .$$

Letting $\eta = \frac{\xi}{\|\xi\|'}$ implies that

$$\|\xi\|' \geq \|((\xi | \, \xi))\| \, \frac{1}{\|\xi\|'} \implies \|\xi\|' \geq \|\xi\|_o \, .$$

Conversely this also works for all vectors η which implies

$$\begin{aligned} \|\xi\|' &= \sup\left\{\|((\eta | \, \xi))\| \, : \, \|\eta\|' \leq 1\right\} \\ &\leq \sup\left\{\|((\eta | \, \xi))\| \, : \, \|\eta\|_o \leq 1\right\} \\ &= \|((\xi | \, \xi))\|^{\frac{1}{2}} = \|\xi\|_o \, . \end{aligned}$$

Thus, $\|\cdot\|_o$ is the unique operator space matrix norm for which the conjugate Hilbert operator space and the dual Hilbert operator space are completely isometric.

□

References

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